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EMPIRICAL FORMULÆ FOR APPROXIMATE COMPUTATION.

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[NOTE. The following paper is here printed in the exact form in which it was left by the author at the time of his death, except only a few necessary verbal alterations which are distinguished from the other parts of the paper by an enclosure of square brackets.]

The following symbols have the same signification, throughout these formulae, the only exception being when the letter "d" is used with a, b, c, etc.

$n = \frac{1}{m}$ = index of root.

$m = \frac{1}{n}$ = index of power.

q = quantity, or $\frac{q}{p}$, if fractional.

s = sum of $q+p$ (or the semi-sum).*

d = difference of q and p (or the demi-difference).*

$t = \frac{n^2-1}{3}$.

U = the sub-square, or under-square $= (\sqrt{q}-1)^2$.

U_4 = the under-fourth $= (\sqrt[4]{q}-1)^4$, etc.

E = Napierian logarithm of number.

K = logarithm of number to base 4.

r = root of number.

*These last values for s and d cannot be used in the same equation with p and q.

$=|=$ signifies nearly equal.

$=\|=$ signifies very nearly equal.

Mercator's formula for the extraction of roots of numbers near unity was equivalent to my own formula No. 1, as shown in Hutton's *Tracts on Mathematics*, Vol. I.

$$\frac{ns+d}{ns-d}=\sqrt[n]{q}, \text{ nearly} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hutton says he gave a formula for the correction of this result, but I have never been able to find it. Hutton himself gives the derivation of the above formula.

First correction of above:

$$\sqrt[n]{q}=|=\frac{ns+d-\dfrac{(n^2-1)d^2}{3ns}}{ns-d-\dfrac{(n^2-1)d^2}{3ns}}.$$

Substituting t for $\dfrac{n^2-1}{3}$, we have

$$\sqrt[n]{q}=|=\frac{ns+d-\dfrac{td^2}{ns}}{ns-d-\dfrac{td^2}{ns}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Second correction of above:

$$\sqrt[n]{\frac{q}{p}}=|=\frac{ns+d-\dfrac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}{ns-d-\dfrac{t(d^2+[\sqrt{q}-\sqrt{p}]^4)}{ns}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

or by reducing,

$$\sqrt[n]{q}=|=\frac{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}+3nd}{(n^2+2)s+(2n^2-2)\sqrt{s^2-d^2}-3nd} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This nearly equals

$$\sqrt[n]{q}=|=\frac{ns+d-\dfrac{n^2-1}{3}\times\dfrac{d^2}{s^2-\dfrac{d^2}{4s}}}{ns-d-\dfrac{n^2-1}{3}\times\dfrac{d^2}{s^2-\dfrac{d^2}{4s}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Simpler than these, and nearly related to (3) is

$$\sqrt[n]{q} = \cfrac{ns + d - \cfrac{td^2}{ns - \cfrac{td^2}{ns}}}{ns - d - \cfrac{td^2}{ns - \cfrac{td^2}{ns}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

Very simple and excellent is

$$\sqrt[n]{q} = \cfrac{ns + d - \cfrac{2tU}{n}}{ns - d - \cfrac{2tU}{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

All the foregoing are what I call *equidistant* processes, because for all values of n, the difference between the numerator and denominator of the result is $2d$, (or some multiple); that is, the subtractive corrections following d are the same in both terms, whether q/p be a proper or an improper fraction.

Of the same nature, but differently derived, is formula (8) and its equivalent (8½).

$$\sqrt[n]{q} = \frac{n(2s^2 - d^2) + d^2 + 2ds}{n(2s^2 - d^2) + d^2 - 2ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

$$\sqrt[n]{q} = \frac{n(2s^2 - d^2) + d(2s + d)}{n(2s^2 - d^2) - d(2s - d)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8\tfrac{1}{2}).$$

The fact that all these fractional formulae are symmetrical makes their operation comparatively simple.

In extracting roots of high numbers

$$\sqrt[n]{q} = 2^{\frac{m}{n}} \times \sqrt[n]{\frac{q}{2^m}},$$

and we are thus always enabled to use $q/2^m$ between the limits ½ and 2. Hence, the following equation becomes of value:

$$\sqrt[n/m]{2} = \frac{3n + m - \dfrac{n^2 - m^2}{9n}}{3n - m - \dfrac{n^2 - m^2}{9n}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\text{or } {}^{n/m}\!\surd 2 = \cfrac{3n+m-\cfrac{n^2-m^2}{9n-\cfrac{n^2-m^2}{3n}}}{3n-m-\cfrac{n^2-m^2}{9n-\cfrac{n^2-m^2}{3n}}}, \quad \ldots\ldots\ldots\ldots (10),$$

$$\text{or } {}^{n/m}\!\surd 2 = \cfrac{3n+m-\cfrac{29(n^2-m^2)}{507n}}{3n-m-\cfrac{29(n^2-m^2)}{507n}}, \quad \ldots\ldots\ldots\ldots (11).$$

In a latent form, the equidistant principle is also present in the following:

$$\sqrt[n]{\frac{q}{p}} = \frac{(2n^2+3n+1)q^2+(8n^2-2)qp+(2n^2-3n+1)p^2}{(2n^2-3n+1)q^2+(8n^2-2)qp+(2n^2+3n+1)p^2} \ldots\ldots (13).$$

In some of the following applications p is taken $=1$.
This [viz. (13)] becomes:

for ${}^{2}\!\surd q$ .... $\dfrac{5q^2+10q+1}{q^2+10q+5}$ ..... (13—2)

for ${}^{3}\!\surd q$ ..... $\dfrac{14q^2+35q+5}{5q^2+35q+14}$ ..... (13—3)

for $\sqrt[4]{\dfrac{q}{p}}$ ..... $\dfrac{15q^2+42qp+7p^2}{7q^2+42qp+15p^2}$ ..... (13—4)

for ${}^{5}\!\surd q$ ..... $\dfrac{11q^2+33q+6}{6q^2+33q+11}$ ..... (13—5)

for ${}^{6}\!\surd q$ ..... $\dfrac{91q^2+286q+55}{55q^2+286q+91}$ ..... (13—6)

for $\sqrt[7]{\dfrac{q}{p}}$ ..... $\dfrac{20q^2+65qp+13}{13q^2+65qp+20}$ ..... (13—7)

for ${}^{8}\!\surd q$ ..... $\dfrac{51q^2+170q+35}{35q^2+170q+51}$ ..... (13—8)

for ${}^{9}\!\surd q$ ..... $\dfrac{95q^2+323q+68}{68q^2+323q+95}$ ..... (13—9)

for $\sqrt[10]{\frac{q}{p}}$ . . . . . $\frac{77q^2+266qp+57p^2}{57q^2+266qp+77p^2}$ . . . . . (13—10).

for $\sqrt[100]{q}$ . . . . . $\frac{6767q^2+26666q+6567}{6567q^2+26666q+6767}$ . . . . . (13—100).

For very high indices use, without sensible error,

$$\sqrt[n]{\frac{q}{p}} = \frac{(2n+3)q^2+8npq+(2n-3)p^2}{(2n-3)q^2+8npq+(2n+3)p^2} \quad \ldots\ldots\ldots\ldots\ldots (14).$$

(14) is the equivalent of (2).

Upon the logarithmic function depend the following formulæ:

$$\sqrt[n]{q} = \frac{n+\frac{E}{2}+\left(\frac{E}{6}\times\frac{r-1}{r+1}\right)}{n-\frac{E}{2}+\left(\frac{E}{6}\times\frac{r-1}{r+1}\right)} \quad \ldots\ldots\ldots\ldots\ldots (15).$$

$$\sqrt[n]{q} = \frac{3+\frac{2E}{n}+\frac{E^2}{2n^2}}{3-\frac{E}{n}} \quad \ldots\ldots\ldots\ldots\ldots (16).$$

E, the logarithm of $q = \frac{3(q^2-1)}{(q+1)^2+2q}$ . . . . . (17),

or, if q be fractional, $= \frac{3(q^2-p^2)}{(q+p)^2+2qp}$ . . . . . (18),

or, in terms of d and $s = \frac{6ds}{3s^2-d^2}$ . . . . . (19).

This value of E, if q be between .9 and 1.1, is true to the seventh decimal, but may be corrected with very great accuracy, even up to the ninth or tenth decimal, by adding to the result the $\frac{4d^2}{45s^2}$th part of itself.

If, however q be so great as

1.7 or so small as .6 use 44 in place of 45
1.8 or so small as .56 use 43 in place of 45
1.9 or so small as .53 use 42 in place of 45
2. or so small as .5 use 41 in place of 45.

The accuracy of these formulæ will appear from the natural logarithms in the next [paragraph].

[*Some verifications of formula* (18):

For $q/p=\frac{11}{10}$:—

By the formula: $\log(q/p)=\frac{63}{661}=.0953101$,
error=1 in last place.

By the tables: 2.3978951
2.3025851
$\log(q/p)=$ .0953100

For $q/p=\frac{16}{15}$:—

By the formula: $\log(q/p)=\frac{923}{1441}=.0645385$,
correct to last place.

By the tables: 2.7725887
2.7080502
.0645385

For $q/p=\frac{301}{100}$:—

By the formula: $\log(q/p)=\frac{6063}{60601}=.0099503$3.]

Reconverting this by (15) we have

$$\frac{1+.004975165+.000008251}{1-.004975165+.000008251}=\frac{1.004983416}{.995033086}.$$

To this denominator add the 100th part,

.995033086
.009950331
1.004983417

Hence the real number is $\frac{101}{100}$.

From the foregoing we have another value of ${}^{n}\!/_{1}\,q$ as follows:

$${}^{n}\!/_{1}\,q=\frac{2n+\left(3+\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)}{2n-\left(3-\frac{r-1}{r+1}\right)\left(\frac{q^2-p^2}{q^2+4qp+p^2}\right)} \quad \ldots\ldots\ldots\ldots\ldots (20).$$

(Of course r can only be taken crudely, but may by successive steps, until the required approximation is reached.)

Another formula akin to (15), for values of q (or r) in terms of E is

$${}^{n}\!/_{1}\,q=\frac{n+\frac{E}{2}+\frac{E^2}{12}}{n-\frac{E}{2}+\frac{E^2}{12}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15\tfrac{1}{2}).$$

Let us call c, which is equal to $\frac{q-1}{E}$, a *root centre*, meaning thereby that, for all values of q, and degrees of n, $\frac{c+\frac{q-1}{q+1}}{c-\frac{q-1}{q+1}}$ nearly equals ${}^{n}\sqrt{q}$. Then

$$\frac{nc+\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}{nc-\frac{q-1}{2}+\frac{(q-1)(r-1)}{6(r+1)}}=|={}^{n}\sqrt{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

c, if q is under 10, is nearly $\frac{2\sqrt{q}+\frac{q+1}{2}}{3}$ .... (22).

c, if q is between 50 and 250, is nearly same $-\frac{q^{\frac{3}{4}}}{500}$ ....(23).

c, if q is between 10 and 50 [correction not given].

FACTOR PROCESS.

Take $\frac{b}{a}\times\frac{c}{b}\times\ldots\ldots\ldots\times\frac{h}{g}=q$; then $\frac{h}{a}=q$ and the series is consecutive.

Take $\frac{b}{a}\times\frac{d}{c}\times\ldots\ldots\ldots\times\frac{h}{g}=q$; then $\frac{bd\ldots.h}{ae\ldots\ g}=q$.

Now if, to take the nth root of q, we assume n terms, consecutive or non-consecutive, and nearly equal in value

$${}^{n}\sqrt{q}=\frac{\frac{\sqrt{b}}{\sqrt{a}}+\frac{\sqrt{d}}{\sqrt{c}}+\ldots\ldots\ldots+\frac{\sqrt{h}}{\sqrt{g}}}{\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{c}}{\sqrt{d}}+\ldots\ldots\ldots+\frac{\sqrt{g}}{\sqrt{h}}}, \quad \ldots\ldots\ \ldots\ldots (24),$$

wherein the numerators are the square roots of the terms and the denominators the reciprocals thereof.

If the terms are consecutive and odd in number

$${}^{n}\sqrt{q}=\frac{ah+2bg+\ldots\ldots\ldots+2de}{2ag+2bf+\ldots\ldots\ldots d^{2}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

but if consecutive and even in number

$$\sqrt[n]{q} = \frac{ai + 2bh + \ldots\ldots\ldots + e^2}{2ah + 2bg + \ldots\ldots\ldots + 2de} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25\tfrac{1}{2}).$$

Also if the series is consecutive,

$$\sqrt[n]{q} = \frac{(g+h)b + (f+g)c + \ldots\ldots\ldots + (b+c)g + (a+b)h}{(g+h)a + (f+g)b + \ldots\ldots\ldots + (b+c)f + (a+b)h}.$$

(25) and (25½) are called the *diagonal* process.

To separate a quantity q/p into consecutive factors which shall be nearly equal, and which shall be as many as there are units in the index of the root to be extracted, and whose differences shall also be in arithmetical progression, use for expansor $\frac{n^2 s}{d^2}$, and then arrange terms with differences themselves differing by unity. This may be done by dividing d', the difference of the expanded fraction q'/p', or $q' - p'$, by n, and making the final interval $\frac{n-1}{2}$ less than the quotient, —after which ascend accordingly.

Illustrations: 1. Separate $\frac{q}{1}$ into three factors of above nature:—

$$\frac{n^2 s}{d^2} = 13.5. \quad \text{Then } \frac{q'}{p'} = \frac{27}{13.5}, \quad \frac{13.5}{3} = 4.5, \quad \frac{n-1}{2} = 1.$$

Then first interval=3.5. Therefore:

$$\frac{17}{13.5}, \quad \frac{21.5}{17}, \quad \frac{27}{21.5}, \text{ or } \frac{34}{27}, \quad \frac{43}{34}, \quad \frac{54}{43}$$

are the desired terms.

Applying the diagonal process:

$$\frac{1458 + 1462 + 1462}{1161 + 1161 + 1156} = \frac{4382}{3478} = \frac{2191}{1739} = 1.2599195. \quad \text{Error} = .0000015.$$

2. Separate $\frac{5}{3}$ into 4 factors:—
Expansor=16 gives $\frac{5}{3} = \frac{80}{48}$; first interval=6.5.
[The series of consecutive factors is]

$$\frac{54.5}{48}, \quad \frac{62}{54.5}, \quad \frac{70.5}{62}, \quad \frac{80}{70.5}, \text{ or } \frac{109}{96}, \quad \frac{124}{109}, \quad \frac{141}{124}, \quad \frac{160}{141}.$$

3. Separate $\frac{3}{1}$ into five factors:—
Expansor=12.5; first interval=3, [and the series of consecutive factors is]

$$\frac{15.5}{12.5}, \frac{19.5}{15.5}, \frac{24.5}{19.5}, \frac{30.5}{24.5}, \frac{37.5}{30.5}, \text{ or } \frac{31}{25}, \frac{39}{31}, \frac{49}{39}, \frac{61}{49}, \frac{75}{61}.$$

If the factors are consecutive and in arithmetical progression, then

$$\sqrt[n]{q} = \frac{n^2 q + \frac{n^2-1}{6}(q-1)^2}{n^2 q - n(q-1) + \frac{(n-1)(n-2)}{6}(q-1)^2} \quad \ldots\ldots\ldots\ldots (27).$$

All quantities p/q may be represented in n terms, which group in three classes as follows:

$$\left(\frac{b}{a}\right)^j \times \left(\frac{d}{c}\right)^k \times \left(\frac{b}{e}\right)^l = q/p, \text{ where } j+k+l=n.$$

$$\text{Then } \sqrt[n]{q} = \frac{j(fc+de)b + k(fa+be)d + l(ad+bc)f}{j(fc+de)a + k(fa+be)c + l(ad+bc)e} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

The above is called the *three-class* process.

Sometimes a quantity will reasonably resolve into n terms, which group in only two classes. Hence the following *two-class* process:

$$\left(\frac{b}{a}\right)^j \times \left(\frac{d}{c}\right)^k = q;$$

$$\sqrt[n]{q} = \frac{j(ad+bc+2cd)b + k(ad+bc+2ab)d}{j(ad+bc+2cd)a + k(ad+bc+2ab)c} \quad \ldots\ldots\ldots\ldots (29).$$

Or, the following, simpler but not so good:

$$\sqrt[n]{q} = \frac{(c+d)jb + (a+b)kd}{(c+d)ja + (a+b)kc} \quad \ldots\ldots\ldots\ldots\ldots (30).$$

Of all these processes the three-class (28) is the most trust-worthy. When q does not naturally resolve in such terms, take q/pv which does, and extract $\sqrt[n]{v}$, and multiply results.

In the consecutive series

$$\frac{b}{a} \times \frac{c}{b} \times \ldots\ldots\ldots \times \frac{e}{d} \times \frac{b}{e} = q,$$

take a new term, of the first expanded q times, that is qb/qa. Then drop the term b/a, and call $qb=g$ and $qa=f$, giving the equation

$$\frac{c}{b}\times\frac{d}{c}\times\ldots\ldots\times\frac{f}{e}\times\frac{g}{f}=q.$$

Now apply the diagonal process, according to the spirit, and not the letter, of (25), and we have,

$$\frac{bg+2cf+2de}{2bf+2ce+d^2}=|=\sqrt[n]{q}\quad\ldots\ldots\ldots\ldots\ldots\ldots(25\tfrac{1}{2}).$$

Now this result will be found no nearer than the result in (25) but the mean of the two will give a close result. It is not, however, a formula of value, and I think there are cases where the error of (25) and (25½) are both on the *same* side. Hence, their mean would be of no value in particular.

PROCESS FOR SPECIAL ROOTS.

For Square Root: $\left[\text{Hutton gives } \frac{ac+b^2}{2ab}=\sqrt{\frac{b}{a}\times\frac{c}{b}}\right]$.

$$\sqrt[2]{q}=|=\frac{(c+d)b+(a+b)d}{(c+d)a+(a+b)c}\quad\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

$$\sqrt[2]{q}=|=\frac{3bcd+3abd+b^2c+d^2a}{3abc+3acd+a^2d+b^2c}\quad\ldots\ldots\ldots\ldots\ldots(32).$$

$$\sqrt[2]{q}=|=\frac{(\sqrt{cd}\times b)+(\sqrt{ab}.d)}{(\sqrt{cd}\times a)+(\sqrt{ab}\times c)},$$

$$\text{or }=|=\frac{b\sqrt{cd}+d\sqrt{ab}}{a\sqrt{cd}+c\sqrt{ab}}\quad\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

If we call r an approximate value of the required square root, then

$$\frac{q+r}{p+r}=|=\sqrt[2]{\frac{q}{p}}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

$$\text{or }\frac{q^2+6r^2q+r^4}{4r(q+r^2)}=|=\sqrt[2]{q}\quad\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

For Cube Roots:—

$$\frac{7q^3+42q^2+30q+2}{2q^3+30q^2+42q+7}=|=\sqrt[3]{q}\quad\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Also, if $\frac{b}{a}\times\frac{d}{c}\times\frac{f}{e}=q$,

$$\sqrt[3]{q} = \frac{bdc+bef+adf}{acf+ade+bce} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37).$$

This is the best of all cube root processes, and I call it the *interweaving* process. It has remarkable properties.

For the $\frac{3}{2}$ Root :—If $\frac{b}{a}\times\frac{c}{b}\times\frac{d}{c}=q$; then

$$\frac{2bd+c^2}{2ac+b^2} = \sqrt[3]{q}, \text{ or } \sqrt[3]{q^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (38).$$

For the Fourth Root:—Let

$$\frac{b}{a}\times\frac{c}{b}\times\frac{d}{c}\times\frac{e}{d}=q.$$

Then $$\sqrt[4]{q} = \frac{(be+cd)b+(ae+bd)e+(ad+bc)d+(ac+b^2)e}{(be+cd)a+(ae+bd)b+(ad+bc)c+(ac+b^2)d} \quad \ldots\ldots\ldots (39).$$

UNDER-SQUARE FORMULÆ.

$U=(\sqrt{q}-\sqrt{p})^2$, $U_4=(\sqrt[4]{q}-\sqrt[4]{p})^4$, $U_8=(\sqrt[8]{q}-\sqrt[8]{p})^8$ .....

(7) is an under-square formula and, if q is an even square, is precisely equivalent to the $\frac{1}{2}n$th root of $\sqrt{q}$, by the first correction of formula 1.

The first correction of (7) is

$$\sqrt[n]{\frac{q}{p}} = \frac{ns+d-\frac{2(n^2-1)}{3n}U-\frac{n^2-4}{6n}U_4}{ns-d-\frac{2(n^2-1)}{n}U-\frac{n^2-4}{6n}U_4} \quad \ldots\ldots\ldots (40).$$

Also, $$\sqrt[n]{\frac{q}{p}} = \frac{ns+d-\frac{2(n^2-1)}{3n}U-\frac{(E-1)U}{72}\times\frac{n(n-2)}{n-1}}{ns-d-\frac{2(n^2-1)}{3n}U-\frac{(E-1)U}{72}\times\frac{n(n-2)}{n-1}} \quad \ldots\ldots\ldots (41).$$

Still another but complex value of $\sqrt[n]{\frac{q}{p}}$ is :

$$\frac{ns+d-\frac{(2n^2-n)}{3n}U-\frac{(n^2-4)\,U_4}{6n}-\frac{(n^2-16)(\sqrt[8]{q}-\sqrt[8]{p})^4\times(\sqrt{q}+1)\sqrt{p}}{*12n}}{ns-d-\frac{(2n^2-n)}{3n}U-\frac{(n^2-4)\,U_4}{6n}-\frac{(n^2-16)(\sqrt[8]{q}-\sqrt[8]{p})^4\times(\sqrt{q}+1)\sqrt{p}}{*12n}} \quad \ldots (42).$$

*It seems that $11n$ is better in actual practice.

For cube root use for coefficients $\frac{16}{9}$ and $\frac{5}{19}$ instead of $\frac{16}{9}$ and $\frac{5}{18}$.

Illustrations: All the equidistant formulæ, except these, approach accuracy only when q is near unity. No such restriction binds these [the undersquare] formulæ, especially those which are most developed. And here, let me say, is undoubtedly the beginning of a series which I think the Calculi would unfold, and I trust some friend of science will take the burden of solving it. So developed, I am satisfied that all roots of all numbers would be extractable to any required degree of accuracy.

1. Taking (40) extract $\sqrt[3]{4096}$.

$$\text{Root}=\frac{3\times4097+4095-\frac{16}{9}\times3969-\frac{5}{19}\times2401}{3\times4097-4095-\frac{16}{9}\times3969-\frac{5}{19}\times2401}$$

$$=\frac{12291+4095-7056-667\text{ nearly}}{12291-4095-7056-667\text{ nearly}}=\frac{8663}{473}=18+;$$

but taking $\frac{5}{19}$ instead of $\frac{5}{18}=\frac{8696}{508}=17+$.

2. Taking $\sqrt[4]{4096}$, we have

$$\text{Root}=\frac{4\times4097+4095-\frac{5}{2}\times3969-\frac{1}{2}\times2401}{4\times4097-4095-\frac{5}{2}\times3969-\frac{1}{2}\times2401}=\frac{9360}{1170}=8.$$

3. Take $\sqrt[6]{4096}$. By (42),

$$\text{Root}=\frac{6\times4097+4095-\frac{35}{9}\times3969-\frac{8}{9}\times2401-\frac{5}{18}\times\frac{100}{9}\times65}{6\times4097-4095-\frac{35}{9}+3969-\frac{8}{9}\times2401-\frac{5}{18}\times\frac{100}{9}\times65}=\frac{11907.2}{3717}=3.2,$$

but should=4.

To Sum a Series: $S=1, r, r^2, \ldots\ldots r^{n-1}$, n terms.

Let $s=\text{ratio}+1=r+1$, $d=r-1$.

$$S=\frac{2n}{s-nd+\frac{(n^2-1)d^2}{3s}}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43).$$

If s exceeds 2, this is not accurate enough to be of value.

Cube Root by Difference Method.

Take $a^3<q$, $b^3>q$. Call $q-a^3=A$, $b^3-q=B$. Then

$$\frac{a^2B+b^2A}{aB+bA}=\sqrt[3]{q}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44).$$

CONVENIENT FORMULA FOR ROOTS OF 2.

$$\sqrt[n]{2} = \frac{101n+35+\frac{35(r-1)}{6(r+1)}}{101n-35+\frac{35(r-1)}{6(r+1)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45).$$

A ROUGH VALUE OF q IN TERMS OF E.

$$q = \frac{\sqrt{9+3E^2}+2E}{3-E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46).$$

LOGARITHMS.

A curious fact, but scarcely a useful one, is: The logarithm of any number approaches

$$K_B = \frac{\sqrt{B\times(q-B^m)}}{\sqrt{B\times(\sqrt{B}-1)B^m+(\sqrt{B}-1)q}}+m, \quad \ldots\ldots\ldots (47),$$

wherein K is the logarithm, B the base of the system, m the characteristic of the logarithm of q, the quantity. Now if $B=4$, the formula becomes

$$K_4 = \frac{2(q-B^m)}{2B^m+q}+m, \quad \ldots\ldots\ldots\ldots\ldots\ldots (48).$$

If $q=q/p$, then $K_4 = \frac{2q-2B^m p}{2B^m p+q}+m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49).$

If $K_4 = \frac{m+c}{d}$, then $q = B^m\left(1+\frac{3c}{2d-c}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$

$$= B^m\frac{2d+2c}{2d-c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (51).$$

Since Napierian $\log E = \frac{44}{61}K$, and also, since

$$\sqrt[n]{q} = \frac{\frac{44}{61}K+\frac{1}{2n}}{\frac{44}{61}K-\frac{1}{2n}}$$

roughly, then

$$^{n}\!/q = \frac{\frac{44}{61}K + \frac{1}{2n}}{\frac{44}{61}K - \frac{1}{2n}} \text{ roughly} \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

The term K also may be understood to mean the logarithm of q to base 4.

K of $10 = \frac{5}{3}$; then $q = 4(1+\frac{6}{4}) = 10$.

The error by this method is easily represented by a curve.

In the fractional processes continually occur the coaddition of fractions, or the adding of numerators together, and of denominators also, when the latter are not the same. When the fractions, two or more, are "embracing," the results are close, and if "harmoniously embracing" then positively accurate geometric averages. They are harmonious when the product of the two terms of each fraction are the same.

Thus, $\frac{8}{6}$ and $\frac{7}{6}$ are embracing, and their co-sum doubled is $2 \times 1\frac{5}{1} = 2.727+$. The real sum is $\frac{8}{3}\frac{3}{0} = 2.767+$. $1\frac{2}{7}$, $1\frac{1}{8}$ and $1\frac{0}{9}$ are embracing, and their co-sum tripled is $3 \times \frac{3}{2}\frac{3}{4} = 4.125$. The real sum is $\frac{2}{5}\frac{1}{0}\frac{1}{4}\frac{7}{7} = 4.200$. $\frac{9}{4}$ and $\frac{6}{6}$ are harmoniously embracing. Their co-sum is $1\frac{5}{1}\frac{5}{0}$ or $\frac{3}{2}$ which is their square root and precise geometric average.

Rough addition may be performed by co-addition. Thus, $1\frac{0}{1}\frac{9}{7} + \frac{2}{3}\frac{1}{3} = \frac{2}{3}\frac{1}{3}$, or $_{1}^{7}\!/_{1}$, which doubled differs from $\frac{3}{2}\frac{4}{7}\frac{3}{2}$ as 3817 differs from 3808, or 1 in 423 parts.

In the foregoing processes for extraction of roots, calculation of logarithms, etc., excepting the factor-process, the accuracy increases rapidly as q approaches unity. In general they have value only when q lies between .5 and 2. Their value should be tested by logarithms. When q exceeds 2, it may generally be divided by the nth power of some simple quantity, which will bring it near unity. If this be difficult, use the nearest power of 2 (say m) as previously explained.

Many apparently absurd problems are readily answered by these formulæ. Thus: *If the cube root of* 2.5 $= 1\frac{9}{1}\frac{9}{4}$ *what is the 4th root of* 3.333+?

Using the 2-class process, formula (29),—

$$\left(\frac{19}{14}\right)^{3} \times \left(\frac{4}{3}\right) = (\text{say to } 3\tfrac{1}{3}) \frac{(125\times3\times19)+(645\times4)}{(375\times14)+(645\times3)}$$

$$= \frac{9}{1}\frac{7}{1}\frac{9}{6}\frac{5}{0} = 1\frac{9}{1}\frac{4}{4}\frac{1}{3}\frac{1}{2} = 1.35545, \text{ [error here]}$$

or by (30),

$$\frac{399+132}{294+99} = \frac{531}{393} = \frac{177}{131} = 1.35114.$$

True answer = 1.35120.

Comparison of Fraction.

It is frequently necessary, or desirable to compare the values of two unwieldly, yet not very unequal fractions, or to ascertain approximately the comparative value of two ordinary fractions quickly, even if only approximately. Hence the following:

Compare b/a with d/c. Suppose them nearly equal to z/y, and divide each fraction by it, giving by/az and dy/cz. Subtract unity from each, and we have $\frac{by \pm az}{az}$ which make equal to $\frac{1}{uaz}$, and $\frac{dy-cz}{cz}$ which make equal to $\frac{1}{vcz}$. Then

$$\frac{uaz \mp vcz}{(uaz) \times (vcz)} = \frac{(ua \mp vc)z}{uavcz}$$

=the relative excess of b/a over d/c. If u and v are each unity, the process is very simple.

Illustrations: 1. Compare $\frac{5}{8}$ and $\frac{7}{11}$ with $\frac{2}{3}$ as measure.

$$(\tfrac{5}{8} \div \tfrac{2}{3}) - 1 = -\tfrac{1}{16} \text{ and } (\tfrac{7}{11} \div \tfrac{2}{3}) - 1 = -\tfrac{1}{22}.$$

$$\frac{22-16}{22 \times 16} = \frac{1}{58\frac{2}{3}}, \text{ or } = \frac{11-8}{11 \times 8 \times 2} = \frac{1}{58\frac{2}{3}}.$$

True answer=$\frac{1}{56}$.

2. Compare $\frac{3}{2}\frac{9}{5}$ with $\frac{6}{4}\frac{5}{9}$. Take base=$\frac{4}{3}$.

$$\frac{by}{az} = \frac{116}{117}, \quad \frac{dy}{cz} = \frac{196}{195}, \quad \frac{195+117}{195 \times 117} = \frac{312}{22815} = \frac{1}{73.1+}.$$

By the formula $\frac{49+29}{49 \times 29 \times 4} = \frac{76}{5684} = \frac{1}{74.8}$.

True answer=$\frac{1911-1885}{1911} = \frac{1}{73.5}$.

New York, 1878—1883.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

71. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

A man owes me \$200 due in 2 years, and I owe him \$100 due in 4 years; when can he pay me \$100 to settle the account equitably, money being worth 6%?

I. Solution by P. S. BERG, Principal of Schools, Larimore, North Dakota; and the PROPOSER.

Let x=the time.

Now, the present worth of \$200 for $(2-x)$ years−the present worth of \$100 for $(4-x)$ years must=\$100.

$\frac{10000}{56-3x}$=present worth of \$200 for $(2-x)$ years at 6%.

$\frac{5000}{62-3x}$=present worth of \$100 for $(4-x)$ years at 6%.

$\therefore \frac{10000}{56-3x}-\frac{5000}{62-3x}=100.$

$\therefore x=.358615$ years=4 months and 9 days.

II. Solution by FREDERIC R. HONEY, New Haven, Connecticut.

The present value of \$1.12 due 2 years hence is \$1.00. Therefore the present value of \$200.00 due in 2 years is \$200÷1.12=\$178.571. The present value of \$1.24 due 4 years hence is \$1.00. Therefore the present value of \$100.00 due 4 years hence is \$100÷1.24=\$80.645. If we deduct \$80.645 from \$178.571 we have \$97.926, the amount due to me at the present time. This sum placed at interest at 6% would yield \$97.926×.06=\$5.876 in 1 year. The difference between \$100.00 and \$97.926 is \$2.074, the interest which must accumulate in order that the sum may become equal to \$1.00. Therefore since the interest \$5.876 accumulates in 1 year, the interest \$2.074 will accumulate in 2.074÷5.876=0.3529 years. Answer.

72. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Though the length of my field is 1-7 longer than my neighbor's, and its quality is 1-9 better, yet as its breadth is 1-4 less, his is worth \$500 more than mine. What is mine worth? *Encyclopedia Britannica.*

Solution by Misses EVA JONES and NEVA CAROTHERS, Senior Pupils of West Point Graded School.

1. $l : l' :: 7 : 8$. 1st condition.

2. $q : q' :: 9 : 10$. 2nd condition.

3. $b : b' :: 4 : 3$. 3rd condition.

4. $v : v' :: 21 : 21$, multiplying and reducing, and remembering that the value $\propto l.b.q$.

5. Also $v - v' = \$500$. Whence,

6. $v = \$10500$. From (4) and (5),

7. $v' = \$10000$.

This problem was also solved by *B. F. SINE, NELSON S. RORAY, P. S. BERG, F. M. McGAW, J. C. CORBIN, COOPER D. SCHMITT, FREDERIC R. HONEY, H. C. WILKES*, and *G. B. M. ZERR*.

M. A. Gruber sent in a solution of Problem 70, Department of Arithmetic, too late for credit in last issue. His answer is 6.48 years.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Sum to n terms the series, $n\cos\theta + (n-1)\cos 2\theta + (n-2)\cos 3\theta$, etc.

[*Chrystal's Algebra.*]

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

Let $S = n\cos\theta + (n-1)\cos 2\theta + (n-2)\cos 3\theta \ldots\ldots$ Also let $S_s = \sin\theta + \sin 2\theta + \sin 3\theta \ldots\ldots$, and $S_c = \cos\theta + \cos 2\theta + \cos 3\theta \ldots\ldots$

$$S = n[\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots] - [\cos 2\theta + 2\cos 3\theta + \ldots\ldots],$$

$$= (n+1)[\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots] - [\cos\theta + 2\cos 2\theta + 3\cos 3\theta \ldots\ldots],$$

$$= (n+1)S_c - dS_s/d\theta.$$

Now $S_c = [\cos\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$, and $S_s = [\sin\{\frac{1}{2}(n+1).\theta\}\sin\frac{1}{2}(n\theta)]/\sin\frac{1}{2}\theta$.

$$\therefore S = (n+1)\frac{\cos\{\frac{1}{2}(n+1)\theta\}\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta} - \frac{d}{d\theta}\left[\frac{\sin\frac{1}{2}(n+1).\theta\sin\frac{1}{2}(n\theta)}{\sin\frac{1}{2}\theta}\right],$$

probably as compact a form as can be obtained.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let S = sum required,

$$2\sin\tfrac{1}{2}\theta\cos n\theta = \sin\{\theta + \frac{2n-1}{2}\theta\} - \sin\{\theta + \frac{2n-3}{2}\theta\}$$

$$4\sin\tfrac{1}{2}\theta\cos(n-1)\theta = 2\sin\{\theta+\frac{2n-3}{2}\theta\}-2\sin\{\theta+\frac{2n-5}{2}\theta\}$$

$$6\sin\tfrac{1}{2}\theta\cos(n-2)\theta = 3\sin\{\theta+\frac{2n-5}{2}\theta\}-3\sin\{\theta+\frac{2n-7}{2}\theta\}$$

$$8\sin\tfrac{1}{2}\theta\cos(n-3)\theta = 4\sin\{\theta+\frac{2n-7}{2}\theta\}-4\sin\{\theta+\frac{2n-9}{2}\theta\}$$

. . . . . .

$$2n\sin\tfrac{1}{2}\theta\cos\theta = n\sin(\theta+\tfrac{1}{2}\theta)-n\sin(\theta-\tfrac{1}{2}\theta).$$

Adding we get

$$2S\sin\tfrac{1}{2}\theta = (\sin\tfrac{3}{2}\theta+\sin\tfrac{5}{2}\theta+\sin\tfrac{7}{2}\theta+\ldots\ldots+\sin\frac{2n+1}{2}\theta)-n\sin\tfrac{1}{2}\theta,$$

$$= [\sin(\frac{n+2}{2})\sin\tfrac{1}{2}(n\theta)]/\sin\tfrac{1}{2}\theta-n\sin\tfrac{1}{2}\theta.$$

$$\therefore\ S = [\sin(\frac{n+2}{2}\theta)\sin\tfrac{1}{2}(n\theta)-n\sin^2\tfrac{1}{2}\theta]/2\sin^2\tfrac{1}{2}\theta.$$

The series in parenthesis above is summed in all trigonometries in the series, $\sin\alpha+\sin(\alpha+\beta)+$, etc., by making $\alpha=\frac{3}{2}\theta$, $\beta=\theta$.

III. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The given series may be broken up into:

$$n[\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\cos n\theta]$$

$$-[\cos2\theta+2\cos3\theta+3\cos4\theta+\ldots\ldots(n-1)\cos n\theta].$$

To sum the series $\cos\theta+\cos2\theta+\cos3\theta+\ldots\ldots\cos n\theta$, we have

$$\sin\tfrac{1}{2}\theta-\sin\tfrac{3}{2}\theta = -2\cos\theta\sin\tfrac{1}{2}\theta.$$

$$\sin\tfrac{3}{2}\theta-\sin\tfrac{5}{2}\theta = -2\cos2\theta\sin\tfrac{1}{2}\theta.$$

$$\sin\tfrac{5}{2}\theta-\sin\tfrac{7}{2}\theta = -2\cos3\theta\sin\tfrac{1}{2}\theta.$$

. . . . . .

$$\sin\tfrac{1}{2}(2n-1)\theta-\sin\tfrac{1}{2}(2n+1)\theta = -2\cos n\theta\sin\tfrac{1}{2}\theta.$$

Adding, we have, $\sin\tfrac{1}{2}\theta-\sin\tfrac{1}{2}(2n+1)\theta = -2\sin\tfrac{1}{2}\theta\Sigma(n\theta)$.

$$\therefore\ \Sigma(n\theta) = [\sin\tfrac{1}{2}(2n+1)\theta-\sin\tfrac{1}{2}\theta]/2\sin\tfrac{1}{2}\theta = [\cos\tfrac{1}{2}(n+1)\theta\sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta.$$

$$\therefore\ n\Sigma(n\theta) = [n\cos\tfrac{1}{2}(n+1)\theta\sin\tfrac{1}{2}n\theta]/\sin\tfrac{1}{2}\theta.$$

To sum the second part, we have,

$$x^2+2x^3+3x^4+\ldots\ldots(n-1)x^n = [x^2-nx^{n+1}+(n-1)x^{n+2}]/(1-x)^2.$$

Putting $x=\cos\theta+i\sin\theta$, and employing the formula $(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$, we obtain after putting the real parts of both members equal, and making all necessary reductions, for the sum of the second series

$$=\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta};$$

so that the sum of the given series

$$=\frac{n\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}+\frac{\cos\theta-n\cos n\theta+(n-1)\cos(n+1)\theta}{4\sin^2\frac{1}{2}\theta}.$$

To test this formula we must of course, leave the coefficient n of the first expression unchanged, while in all the other factors and terms which involve n, n must be put successively=1, 2, 3, 4, etc.

Also solved by *E. W. MORRELL*.

69. Proposed by C. E. WHITE, A. M., Trafalgar, Indiana.

Prove that $x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}\pm\ldots\ldots\ldots+(\pm1)^nx$, where A, B, C, ... are the binomial coefficients of the $(n+1)$th order.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

$$x^n\pm x^{n-1}+x^{n-2}+\ldots\ldots\ldots, \text{ etc.}=(x^{n+1}-1)/(x-1)\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $x^{n+1}+1)/(x+1)\ldots\ldots\ldots\ldots\ldots\ldots(2)$, $=\{[(x-1)+1]^{n+1}-1\}/(x-1)$, or

$\{[(x+1)-1]^{n+1}+1\}/(x+1)$, $=(x-1)^n+C^n_{n+1}(x-1)^{n-1}+C^n_{n+1}(x-1)^{n-2}+\ldots\ldots$,

$$\text{or } (x+1)^n-C^n_{n+1}(x+1)^{n-1}+C^n_{n+1}(x-1)^{n-2}-\ldots\ldots,$$

$$=(x\mp1)^n\pm A(x\mp1)^{n-1}+B(x\mp1)^{n-2}+\ldots\ldots$$

II. Solution by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

Let $K=x^n\pm x^{n-1}+x^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}x+(\pm1)^n$.

Put $x=y\pm1$, expanding and observing that the sign of the last term of each expression is $\pm$ if n is odd but $+$ if n is even, we may write:

$x^n=(y\pm1)^n=y^n\pm ny^{n-1}+\frac{1}{2}[n(n-2)]y^{n-2}\pm\ldots\ldots\ldots\ldots\ldots+(\pm1)^{n-1}ny+(\pm1)^n$

$\pm x^{n-1}=\pm(y\pm1)^{n-1}=\pm y^{n-1}+(n-1)y^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}(n-1)y+(\pm1)^n$

$x^{n-2}=(y\pm1)^{n-2}=\ldots\ldots\ldots\ldots\ldots y^{n-2}\pm\ldots\ldots\ldots+(\pm1)^{n-1}(n-2)y+(\pm1)^n$

etc ..... etc.

$(\pm 1)^{n-1}x=(\pm 1)^{n-1}(y\pm 1)=$ .... $(\pm 1)^{n-1}y+(\pm 1)^n$

$(\pm 1)^n=$..... ..... $(\pm 1)^n$.

By adding, and simplifying the coefficients of y, we have

$K=y^n\pm(n+1)y^{n-1}+\frac{1}{2}[(n+1)n]y^{n-2}\pm$ .... $+(\pm 1)^{n-1}\frac{1}{2}[(n+1)n]y+(\pm 1)^n(n+1)$,

which has binomial coefficients of the $(n+1)$th order. Substituting A, B, C, ... for the coefficients and restoring the values of y,

$K=(x\mp 1)^n\pm A(x\mp 1)^{n-1}+B(x\mp 1)^{n-2}\pm$ ..... $+(\pm 1)^{n-1}B(x\mp 1)+(\pm 1)^nA$.

[Expanding and combining the terms of the second member, we get the first member for a result. ZERR.]

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

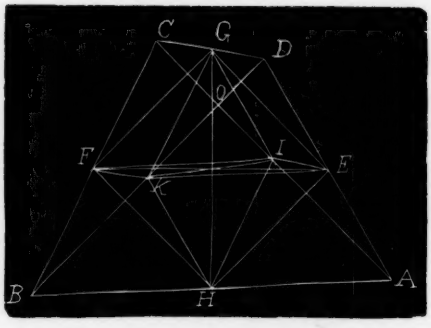
SOLUTIONS OF PROBLEMS.

43. Proposed by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

The consecutive sides of a quadrilateral are a, b, c, d. Supposing its diagonals to be equal, find them and also the area of the quadrilateral.

II. Solution by A. H. BELL, Hillsboro, Illinois.

The solution as published simply demonstrates this theorem, $a^2+b^2+c^2+d^2=2x^2+4\overline{IK}^2$, with two unknowns, and is then solved for a particular case.



Let the sides of the quadrilateral AB, BC, CD, and AD, be a, b, c, and d; and the diagonals each $=2x$; $x+y$, $x-y$=the segments AO and OC; and BO and $OD=x+z$ and $x-z$. In the triangles AOB, BOC, and COD we have $(x+y)\cos A+(x+z)\cos B=a$, $\cos A=(4x^2+a^2-b^2)/(4ax)$, $\cos B=(4x^2+a^2-d^2)/(4ax)$, making

$z(4x^2+a^2-d^2)=(2a^2+b^2+d^2)x-8x^3-(4x^2+a^2-b^2)y$ .....(1).

Similarly BOC, $z(4x^2+b^2-c^2)=(2b^2+a^2+c^2)x-8x^3-(a^2-b^2-4x^2)y$ .....(2),

and COD, $z(-4x^2+b^2-c^2)=(b^2+2c^2+d^2)x-8x^3+(c^2-d^2+4x^2)y$ .....(3).

Subtracting (2) from (1), $(a^2-b^2+c^2-d^2)z=(a^2-b^2-c^2+d^2)x-8x^2y$ .....(4).

Subtracting (3) from (2), $8x^2z=(a^2+b^2-c^2-d^2)x-(a^2-b^2+c^2-d^2)y$.....(5).

Equating the values of z in (4) and (5) and solving for y, after letting

$(a^2-b^2+c^2-d^2)=e$, $a^2-b^2-c^2+d^2=f$, $a^2+b^2-c^2-d^2=g$, we have

$$y=(eg-8fx^2)/(e^2-64x^4).x \quad \text{.....(6).}$$

Equating the values of z and solving for y in (1) and (4) after letting

$$a^2-b^2=m,\ \ a^2-d^2=n,\ \ 2a^2+b^2+d^2=p,\ \ 2e+f=q,$$

and noting that $2n-e=g$, we have

$$y=(4qx^2-ep+fn)/(32x^4+4gx^2-em).x \quad \text{.....(7).}$$

Equating the values of y in (6) and (7),

$$512x^6-64(a^2+b^2+c^2+d^2)x^4+16[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]x^2$$
$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0 \quad \text{.....(8).}$$

Let $x^2=\frac{1}{8}y$, $4x^2=\frac{1}{2}y$, $2x=\sqrt{\frac{1}{2}y}$, or multiply (8) by the geometrical series, $\frac{1}{8}\frac{1}{2}$, $\frac{1}{8}\frac{1}{4}$, $\frac{1}{8}$, and 1, ratio=8, then (8) becomes

$$y^3-(a^2+b^2+c^2+d^2)y^2+2[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y$$
$$+4(ac-bd)(ac+bd)(a^2-b^2+c^2-d^2)=0 \quad \text{.....(9).}$$

When this is solved by Cardan's formula, then since there are given the sides of the triangles ABC and ACD, we have in the general formula, with the sides a, b, and $2x$, and c, d, and $2x$, the area for the quadrilateral

$$ABCD=\tfrac{1}{4}\sqrt{[(a+b)^2-4x^2]\times[4x^2-(a-b)^2]}$$
$$+\tfrac{1}{4}\sqrt{[(c+d)^2-4x^2]\times[4x^2-(c-d)^2]}.$$

A better transformation for (8) is to put $4x^2=y$, $2x=\sqrt{y}$. We then get $y^3-\frac{1}{2}(a^2+b^2+c^2+d^2)y^2$ $+\frac{1}{4}[a^2(b^2-2c^2+d^2)+b^2(c^2-2d^2)+c^2d^2]y+\frac{1}{2}(ac+bd)(ac-bd)(a^2-b^2+c^2-d^2)=0$.

Example: $a, b, c, d=6, 5, 3, 4$, respectively, in (9) gives

$$y^3-86y^2+894y-1216=0.$$

By Horner's method, $y=75.7270176+$. Diagonal $2x=6.1533+$ agreeing with a close drawing.

[Mr. Bell sent us this solution March 14, 1895. We have looked it over carefully and believe that it is entirely correct. The solution published in the July-August number of Vol. II is of a particular case. EDITOR.]

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

57. Proposed by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation: $(1+x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$

I. Solution by WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let $y=bx\left[\int zdx+c\right]$ and the equation becomes

$$x(1+x^2)\frac{dz}{dx}+2z=0, \text{ or } \frac{dz}{z}+\frac{2dx}{x(1+x^2)}=0.$$

$$\therefore z=c'(1+[1/x^2]);\ y=bx\{c'(x-[1/x])+c\},\ =Bx+A(1-x^2).$$

II. Solution by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pennsylvania.

Proceeding to obtain a solution in series, both values of y are found to terminate immediately. The complete primitive is $y=Ax+B(x^2-1)$.

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

It is shown (*Forsyth's Differential Equations*, Article 58) that

$$d^2y/dx^2+P(dy/dx)+Qy=R \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

gives, when $y=vw$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

$$w\frac{d^2v}{dx^2}+(2\frac{dw}{dx}+Pw)\frac{dv}{dx}+(\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw)v=R \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

with the conditional equations:

$$\frac{d^2w}{dx^2}+P\frac{dw}{dx}+Qw=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$\frac{d^2v}{dx^2}+[(2/w)\frac{dw}{dx}+P]dv/dx=R/w \quad \ldots\ldots\ldots\ldots\ldots (5).$$

w being supposed known from (4) gives

$$w^2\frac{dv}{dx}\varepsilon^{-\int Pdx}=A+\int wR\varepsilon^{\int Pdx}dx \quad \ldots\ldots\ldots\ldots\ldots (6),$$

and $$v=B+A\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}+\int\frac{dx}{w^2}\varepsilon^{-\int Pdx}\int wR\varepsilon^{\int Pdx}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Now (4) is of the same form as (1) excepting that the right member is 0; so that if we have a solution of (4) we have that of (1) when $R=0$.

The given equation is

$$\frac{d^2y}{dx^2}-\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{2}{1+x^2}y=0 \quad \ldots\ldots\ldots\ldots\ldots (8);$$

then $P=-2x/(1+x^2)$, and a particular solution is

$$y=x \quad \ldots\ldots\ldots\ldots\ldots (9), \text{ or } w=x \quad \ldots\ldots\ldots\ldots\ldots (10).$$

Then (7) gives $$v=B-A\int\left(\frac{dx}{x^2}+1\right)=B+A(x-[x/1]),$$

and $y=vx=Bx+A(x^2-1)$, the required solution.

[As will be seen from the last solution both forms are correct. The first form is given as the answer, on page 336 of *Byerly's Integral Calculus*. EDITOR.]

Also solved by *O. W. ANTHONY, W. C. M. BLACK, J. SCHEFFER, G. B. M. ZERR.* and *P. S. BERG*.

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

Solution by the PROPOSER.

Let $y=f(x)$ be the equation of the curve, taking the fixed point as origin. Let (x_1, y_1) be the position of the moving point at time t. Call θ_1 the angle which the line through the fixed point originally makes with axis of x; also, let ω_1 be the rate of angular rotation. Then the equation of line is

$$y=\tan(\omega_1 t+\theta_1)x;$$

whence $t=(\omega_1/1)[\tan^{-1}(y/x)-\theta_1]$ .....(1).

Also the equation of other line is

$$y-y_1=\tan(\omega_2 t+\theta_2)(x-x_1);$$

whence $t=(\omega_2/1)\left(\tan^{-1}\frac{y-y_1}{x-x_1}-\theta_2\right)$ .....(2).

$$ds_1^2=dx_1^2+dy_1^2. \quad ds_1/dt=v_1. \quad \therefore v_1^2=[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}\frac{dx_1^2}{dt^2}.$$

$$t=\frac{1}{v_1}\int[1+\overline{f'(x_1)}^2]^{\frac{1}{2}}\,dx \text{ .....(3).} \quad y_1=f(x_1) \text{ .....(4).}$$

To solve the problem, then, we integrate (3), solve the resulting equation for x_1, substitute this value in (4), and then substitute the values of x_1 and y_1 in (2), after which t is to be eliminated between the resulting equation and (1). To apply this method to the case of the straight line

$$y_1=f(x_1)=0. \quad x_1=v_1 t.$$

$\therefore$ From (2), $t=\frac{1}{\omega_1}\tan^{-1}\left(\frac{y}{x-v_1 t}\right)-\theta_1.$

Then $\frac{1}{\omega_1}\left(\tan^{-1}\frac{y}{x}-\theta_1\right)=\frac{1}{\omega_1}\left(\tan^{-1}\frac{y}{x-(v_1/\omega_1)(\tan^{-1}[y/x]-\theta_1)}-\theta_2\right).$

Let $y=\rho\sin\phi$; $x=\rho\cos\phi$.

Then $\frac{1}{\omega_1}(\phi-\theta_1)=\frac{1}{\omega_2}\left[\tan^{-1}\left(\frac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}\right)-\theta_2\right].$

$$\therefore \frac{\rho\sin\phi}{\rho\cos\phi-(v_1/\omega_1)(\phi-\theta_1)}=\tan\left[\theta_2+\frac{\omega_2}{\omega_1}(\phi-\theta_1)\right].$$

$$\therefore \rho=\frac{(v_1/\omega_1)(\phi-\theta_1)\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]}{\cos\phi\tan[\theta_2+(\omega_2/\omega_1)(\phi-\theta_1)]-\sin\phi},$$

the polar equation of the curve.

The case of the circle leads to complicated results. The case of two fixed points is interesting. The equations of the intersecting straight lines may be written

$y=\tan(\omega_1 t+\theta_1)x$. .....(1), and $y=\tan(\omega_2 t+b_2)(x-a)$.....(2).

From (1) $\tan\phi=\tan(\omega_1 t+\theta_2)$.

$\therefore\ \phi=\omega_1 t+\theta_1.\quad t=(1/\omega_1)(\phi-\theta_1)$.

$\therefore\ \rho\sin\phi=\tan(\omega_2 t+\theta_2)(\rho\cos\phi-a)$;

whence $\rho=[a\tan(\omega_2 t+\theta_2)]/[\cos\phi\tan(\omega_2 t+\theta_2)-\sin\phi]$, or

$$\rho=\{a\tan\,[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]\}/\{\cos\phi\tan[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]-\sin\phi\},$$

$$=\{\,a\sin\,[\frac{\omega_2}{\omega_1}(\phi-\theta_1)+\theta_2]\}/\{\sin[\,(\frac{\omega_2}{\omega_1}-1)\phi+\theta_2-\frac{\omega_2}{\omega_1}\theta_1]\}\ \ldots\ldots\ldots\ldots\ldots\ldots(m).$$

If they both start from the initial horizontal position at the same time,

$$\theta_1=\theta_2=0,\ \text{and}\ \ \rho=a\frac{\sin[(\omega_2/\omega_1)\phi]}{\sin\{[(\omega_2/\omega_1)-1]\phi\}}.$$

A large number of curves is included in equation (m).

This problem was also solved by C. W. M. Black. His solution will appear in the next issue.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a particle placed at a given latitude ?

Solution by the PROPOSER.

Adopt as coördinates the latitude of the particle and the distance measured

in miles along a circle of latitude. Call the latitude λ, and the distance measured along the small circle x. Also let λ_1 be the initial latitude.

As the particle moves towards the equator under the resolved component of centrifugal force there will be no acceleration parallel to the equator.

Then clearly,

$$x=2\pi R(\cos\lambda-\cos\lambda_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also for the acceleration towards the equator,

$$\frac{d^2(R\lambda)}{dt^2}=-\frac{2\pi^2}{T^2}R\sin2\lambda,\quad\text{or}\quad\frac{d^2\lambda}{dt^2}=-\frac{2\pi^2}{T^2}\sin\lambda.$$

Let $d\lambda/dt=\mu$.

Then $\mu=\sqrt{2}\,\frac{\pi}{T}\sqrt{\cos2\lambda-\cos2\lambda_1}$.

Whence $t=\frac{1}{2}\sqrt{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\cos2\lambda-\cos2\lambda_1}}=\frac{1}{2}\frac{T}{\pi}\int_{\lambda_1}^{\lambda}\frac{d\lambda}{\sqrt{\sin^2\lambda_1-\sin^2\lambda}}$.

Let $\sin\lambda=\sin\lambda_1\sin\phi$. Then

$$t=\tfrac{1}{2}\int_{\phi_1}^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}.\quad\phi_1=\tfrac{1}{2}\pi,\quad\phi=\sin^{-1}\Big(\frac{\sin\lambda}{\sin\lambda_1}\Big).$$

$$t=\tfrac{1}{2}\frac{T}{\pi}\Big[\int_0^{\phi_1}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}-\int_0^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\lambda_1\sin^2\phi}}\Big]$$

$$=\tfrac{1}{2}(T/\pi)[F(\sin\lambda_1,\ \tfrac{1}{2}\pi)-F(\sin\lambda_1,\ \phi)]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equation (1) gives the relation between the coördinates at any time, and (2) gives the time of motion.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

Solution by the PROPOSER.

Let R=the radius of desert, T=time, v=rate. Let P be the position of the man at any instant. Draw about P an infinitesimal circle, MSK. Call the angle MPN, θ. Then $\theta=\cos^{-1}\frac{PN}{PM}$.

Now the rate at which the man must approach the circumference in order to be off in a given time is R/T. In an infinitely small time the distance will be $(R/T)dt$.

Also $PN=vdt$.

$$\therefore \theta=\cos^{-1}\left(\frac{R}{Tv}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now R, T, and v are positive. Therefore the value of θ defined by equation (1) has to do with an angle less than 90°.

Now if the man at each instant goes within the angle MPN, he will get off the desert in the given time. The chance that he will do this is

$$C=\frac{2\cos^{-1}[(R/Tv)]}{\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence the required probability in given by (2).

If $R=0$, or T, $=\infty$, or $v=\infty$, $C=1$. If $R=Tv$, $C=0$.

If $R>Tv$, C is impossible.

Let $R=1$, and $v=1$. To find the time which he must have at his disposal in order that he may have half a chance to get off the desert.

Clearly $T=\sqrt{2}$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 feet deep for 540 five-cent loaves when flour is $8 a barrel; what is flour worth a barrel when 45 men, working 5⅓ days of 10 hours each, dig a trench 125 yards long, 5 yards wide, 8 feet deep for 320 four-cent loaves? *Solve by proportion.*

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that the oldest should have one-half, the next one-third, and the youngest one-ninth. His estate, on inventory, was found to consist of 17 elephants. What should be the share of each?

ALGEBRA.

78. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Hewlett, Long Island, New York.

Solve $x^2 + xy = 10 \ldots\ldots (1)$; $y^2 + xy = 15 \ldots\ldots (2)$, for all the roots, and prove that they are the roots.

[Former solutions in print are defective. See *Analyst*, Vol. VIII, page 111; Vol. IX, page 53. J. M. B.]

79. Proposed by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Of n persons A, B, C, etc., A first gives to the others as much as each of them already has; then B gives to the others as much as each then has; and so on for each in turn. Finally, A, B, C, etc., have respectively a, b, c, etc., dollars. How much had each at first?

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Solve $1+x^4 = a(1+x)^4$.

GEOMETRY.

74. Proposed by ROBERT JUDSON ALEY, M. A., Professor of Mathematics in Indiana University, Fellow in Mathematics, University of Pennsylvania.

Let O be the center of the inscribed circle. AO produced meets the circumcircle in A'. Find the ratio of AO to OA'.

75. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

A plane passes through $(0, 0, c)$ and touches the circle $x^2 + y^2 = a^2$, $z=0$; determine the locus of the ultimate intersections of the plane.

76. Proposed by L. B. FRAKER, Bowling Green, Ohio.

Lines run from a point, P, within a triangular piece of land to the angles A, B, and C are 91, 102, and 80 rods, respectively; and a line 78 rods in length passing through the point, P, and terminating in the sides AC and BC cuts off 3024 square rods adjacent to angle C. Required the dimensions of the land.

CALCULUS.

65. Proposed by GEORGE LILLEY, Ph. D., LL. D., Portland, Oregon.

A string is wound spirally 100 times around a cone 100 feet high and 2 feet in diameter at the base. Through what distance will a duck swim in unwinding the string keeping it taut at all times, the cone standing on its base and at right angles to the surface of the water?

66. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Around the top of a conical frustum,—base 5 feet, top 1 foot, altitude 100 feet,—is wound a rope 100 feet long and 1 inch thick. It is unwound by a hawk flying in one plane. How far does Mr. Hawk fly?

67. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, Ohio.

A man starts to walk at a uniform rate across a draw-bridge just as it begins to move. He walks the full length of the bridge and back, in the same time that it takes the bridge to make a half revolution. How far does he ride, the length of the bridge being 250 feet, and its velocity uniform about a center axis?

MECHANICS.

52. Proposed by S. ELMER SLOCUM, Union College, Schenectady, New York.

A chain 16 feet long is hung over a smooth pin with one end 2 feet higher than the other end and then let go. Show that the chain will run off the pin in about 7-5 second. [*Wright's Mechanics*, page 92.]

53. Proposed by J. C. NAGLE, M. A., C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas.

Find the locus of the center of gravity of an arc of constant length for a parabola.

54. Proposed by C. H. WILSON, Poughkeepsie, New York.

A body slides from rest down a series of smooth inclined planes, whose total heights are h feet. Show that the velocity at the bottom is $\sqrt{2gh}$ feet per second. [From *Wright's Mechanics*.]

AVERAGE AND PROBABILITY.

51. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Three points are taken at random in a sphere and a plane passed through them. Find the average volume of the segment cut off from the sphere.

52. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A straight line of length a is divided into three parts by two points taken at random; find the chance that no part is greater than b. [From *Hall and Knight's Higher Algebra*.]

53. Proposed by Samuel E. Harwood, Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Illinois.

Four Latin sentences are given. Number one has 12 words, two has 13 words, three and four have 6 each. What are the chances that two pupils will have them in the same order? Will the result vary with the number of pupils in the class?

EDITORIALS.

President George H. Harter, of Delaware College, Delaware, has just ordered a complete set of the MONTHLY.

We shall be pleased to pay 25 cents each for a limited number of copies of No. 6, Vol. I, and No. 11, Vol. II, of the MONTHLY. Any of our readers wishing to dispose of these numbers should write to us.

We are greatly pleased to note that the Board of City Trusts, Philadelphia, Pennsylvania, has recognized the long and faithful service of Professor Warren Holden in the following resolution: Resolved, That in consideration of forty-five years continued and faithful service, Warren Holden, A. M., Professor of Mathematics at Girard College, be retired January 31, 1897, at a salary of $2,500 per annum.

So far, we have received only a few letters respecting the matter of publishing the portraits of our contributors. We shall be pleased to hear still furthur, and those who favor the plan may send their photos to us at once.

The paper by the late Ansel N. Kellogg, of Chicago, published in this issue was sent to the MONTHLY at the suggestion of Professor Irving Stringham, of the University of California. Professor Stringham says, "They [the formulæ] take us back to methods that were in vogue at the beginning of the century. But they are much superior in accuracy and rapidity of convergence to any I have found in the older books. They will be of some interest, I think, to mathematical readers.

Their author, the late Ansel N. Kellogg, of Chicago, was for a number of years prominent in newspaper and business circles throughout the country. Though a very busy man, he found time for mathematical meditation, and that he could think efficiently in this domain the paper presented sufficiently attests."

As we are very anxious to increase the subscription to the MONTHLY we make the following liberal offers :

1. To any person sending us 75 new subscribers at our regular price, we will make a present of a handsome set of the *Century Dictionary and Encyclopedia*.
2. To any person sending us 50 new subscribers at our regular price, we will make a present of a $100 *Acme* or *Monarch Bicycle*.
3. To any person sending us 20 new subscribers at our regular price, we will make a present of the *Standard American Encyclopedia* [see advertisement on cover.]
4. To any person sending us 15 new subscribers at our regular price, we will make a present of a copy, in one volume, of the *Standard Dictionary* (Funk and Wagnalls').

In all cases the money must accompany the list of names sent in.

BOOKS AND PERIODICALS.

Determinants. Designed for High Schools, and Lower Classes of Colleges and Universities. By J. M. Taylor, M. S., Professor of Mathematics and Astronomy in the University of Washington and Director of the Observatory. 8vo. Cloth, 48 pages. Chicago : Werner School Book Company.

In this little book, Professor Taylor has set forth in a very clear and concise manner the fundamental principles of Determinants. We feel sure that this little work will go far towards popularizing the subject and bringing it within the easy comprehension of the students of our best High Schools. B. F. F.

Elements of Theoretical Physics. By Dr. C. Christiansen, Professor of Physics in the University of Copenhagen. Translated into English by W. F. Magie, Ph. D., Professor of Physics in Princeton University. Large 8vo. Cloth, 338 pages. Price, $3.25. New York : The Macmillan Co.

This work, at first sight, presents a formidable appearance in mathematical notation and formulæ, but by beginning with the introduction and carefully reading through it, the reader is led on to overcome difficulties by a force which can only be accounted for by the admirable, clear, and interesting treatment of the subjects. It presents the fundamental principles of Theoretical Physics and developes them so far as to bring the reader in touch with much of the new work that is now being done in that subject. It is not exhaustive in every respect, but is stimulating and informing and furnishes a view of the whole field, which will facilitate the reader's subsequent progress in special parts of it. The book is printed on good paper and is well bound. Its appearance could have been somewhat improved by not printing it so compactly. B. F. F.

Principles of Mechanism. A treatise on the Modification of Motion by Means of the Elementary Combinations of Mechanism or of the Parts of Machines. For use in College Classes, by Mechanical Engineers, etc., etc. By Stillman W. Robinson, C. E., D. Sc., till recently Professor of Mechanical Engineering in the Ohio State University. First Edition, first thousand. Large 8vo. Cloth, 309 pages. Price, $3.00. New York : John Wiley & Sons.

In this volume we have a thoroughly scientific treatise on mechanical movements. They are treated from the standpoint of both theory and practice. The work embodies the substance of lectures given by the author during the past twenty-seven years.

The work is largely addressed to those who are more conversant with the drawing board than with mathematics, so that the subject has been treated more from the standpoint of graphics than of pure analysis. This feature will popularize the work. The drawings, which are very suggestive, beautiful, and accurate, are very numerous. There are numerous reproductions from actual models. B. F. F.

(1) *Macaulay's Essay on Milton;* (2) *Shakespeare's Midsummer Night's Dream ;* (3) *Scott's Woodstock ;* (4) *Milton's L'Allegro, Il Penseroso, Comus, Lycidas ;* (5) *"George Eliot's" Silas Marner.* Price of (1), (2), and (4) 20 cents, of (3) 60 cents, and of (5) 30 cents. American Book Company, New York, Cincinnati, and Chicago.

We notice collectively this group of texts from the "Eclectic English Classics" series, published by the American Book Company. The texts are well and carefully edited, with introductions and explanatory notes. (1), (3), and (5) have frontispiece portraits of John Milton, Oliver Cromwell, and "George Eliot", respectively. These books are clearly printed, the notes are concise and sufficient, and the introductions interesting and valuable. There is so much advantage in extending the use of these gems of English Literature in our schools, that a debt of gratitude is due the publishers for providing them in such serviceable shape and at a minimum cost. J. M. C.

An Elementary Treatise on Plane Trigonometry. By E. W. Hobson, Sc. D., and C. M. Jessop, M. A. 299 pages. Price, $1.25. Cambridge University Press. New York : Macmillan & Co.

This treatise on trigonometry is a work of recognized merit. The chapter on solution of trigonometrical equations is particularly full and valuable. The plan of the work is good and the execution thorough and satisfactory. J. M. C.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Redpath and Helen H. Gardner. Price, $3.00 per year in advance. Single numbers, 25 cents. Boston : The Arena Company.

The March number of *The Arena* is the initial issue of the magazine under the new management and editorship. The Company has been reorganized on a solid financial basis, and the current number of the magazine comes in a form and substance well calculated to win public favor, and following its well established policy of liberalism and reform.

The number opens with the first of a series of important contributions on the development and reform of city government in the United States. This first article is by the Hon. Josiah Quincy, Mayor of Boston, who therein expresses himself as in favor of the municipal ownership, though not necessarily the municipal operation, of public services, such as gas and electric lighting and street railways. An excellent portrait of Mayor Quincy forms the frontispiece to the number. The article by Professor LeConte, of the University of California, on "The Relation of Biology to Philosophy," is a searching adverse criticism of the seventh chapter of Professor Watson's recent work on "Comte, Mill, and Spencer;" but it is also very much more, being a thoroughly up-to-date exposition of the general theory of organic evolution, and its relation to religion as well as philosophy.

B. F. F.

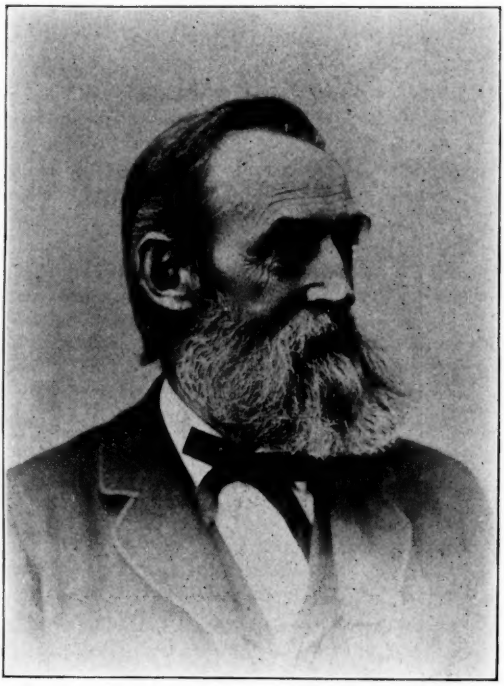
The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

The editor of the *Review of Reviews* comments in the March number on the Spanish program of reforms in Cuba, the United States Senate's attitude toward the arbitration treaty with England, the immigration bill, the proposed international monetary conference, President-elect McKinley's cabinet selections, the recent Senatorial elections, the New York Trust investigation, the famine situation in India, the affair of the Greeks in Crete, the foreign policy of Russia, the position of England, France, and the other great powers, and many other matters of current interest.

B. F. F.

Errata in January Number.

On page 16, 2nd line of problem 55, for "grouhd" read *ground.*
On page 17, in the figure, join *CF* and *CP.*
On page 17, 1st line of solution II, for "*AMFHC*" read *AMFHC'.*
On page 20, line 1, complete the parenthesis after last term of equation.
On page 20, line 8, place — between two terms enclosed by brackets.
On page 20, line 14, for "$\frac{2}{3}\frac{1}{2}\pi^2 a^4$" read $\frac{2}{3}\frac{1}{2}\pi^2 a^4$.
On page 21, line 2, for "$\frac{16}{3}a^2$" read $\frac{16}{3}a^3$.
On page 21, line 8, read $d\rho = 2a\cos^2\theta + a\cos\theta - a = 0$.
On page 25, line 18, for "100" read 100th.
On page 25, line 25, read "add and subtract $B^2/4$, etc."
On page 26, line 15, for "$(2mp)^2$" read $(2mn)^2$.
On page 27, line 3, for "is" read in.
On page 28, line 15, for "$x++(x1)$" read $x+(x+1)$.
On page 28, line 20, for "2392" read 1393.
On page 32, lines 6, 11, and 12, read l where 1 occurs.



HUBERT ANSON NEWTON.